P-ideals and the weak Rudin–Keisler order

Konstantinos A. Beros
and
Paul B. Larson

Miami University

AMS Fall Central Sectional Meeting, Ann Arbor, MI
Tukey order

For partial orders $P$ and $Q$, a Tukey map $f : P \to Q$ is a function which maps $P$-unbounded sets to $Q$-unbounded sets.

Introduced by John Tukey to study convergence in topological spaces.

Isbell and later Fremlin used the Tukey order to study the cofinal structure of interesting ideals (which are partial orders under $\subseteq$):

- $Z_0 = \{X \subseteq \omega : X \text{ has asymptotic density 0} \}$
- $\ell_1 = \{X \subseteq \omega : \sum_{n \in X} 1/(n + 1) < \infty \}$
- $\text{NWD} = \{K \subseteq 2^\omega : K \text{ is closed nwd} \}$
From the work of Isbell, Fremlin, Solecki, Todorcevic and others, the following picture of the Tukey order emerged:

\[
\begin{array}{c}
\text{NWD} \\
\omega^\omega \\
\ell_1 \\
\end{array}
\begin{array}{c}
\rightarrow \\
\nearrow \\
\nearrow \\
\rightarrow \\
\end{array}
\begin{array}{c}
\ell_1 \\
\rightarrow \\
\searrow \\
\zeta_0 \\
\rightarrow \\
\end{array}
\]

The non-reductions turned out to be the hard part.
P-ideals

An ideal $I \subseteq 2^\omega$ is a $P$-ideal iff every sequence in $I$ has a pseudo-union in $I$:

$$(\forall X_0, X_1, \ldots \in I) \ (\exists X \in I) \ (X_n \setminus X \text{ is finite})$$

Examples: $\ell_1$, $Z_0$

Fremlin

The ideal $\ell_1$ is Tukey-maximal among all $\Sigma^1_1$ P-ideals. (Actually he proved it for Polishable lattices – a larger class of partial orders.)
The weak Rudin-Keisler order (K. Beros)

Suppose that $I$ and $J$ are ideals on $\omega$. A $wRK$ map is a function $f : A \to \omega$ such that, for all $X \subseteq \omega$,

$$X \in I \iff f^{-1}[X] \in J.$$ 

In this case, write $I \leq_{wRK} J$. Note: if $I \neq \mathcal{P}(\omega)$, then $A \notin J$.

**Remark**

$I \leq_{wRK} J \iff (\exists A \notin J) (I \leq_{RK} J \cap \mathcal{P}(A))$

Like the Katětov order, the $wRK$ order is a natural weakening of the RK order:

**Proposition**

The $wRK$ order agrees with the RK order on the set of ultrafilters.
Earlier work

K. Beros

Each of the following classes of ideals on $\omega$ has wRK-maximal members:

- $F_\sigma$ ideals
- $\Sigma^1_n$ ideals (uses basic closure properties)
- $\Pi^1_1$ ideals (uses the pwo property)
- $\Pi^1_{2n+1}$ ideals (assuming PD to get the pwo property)
Definable complexity

Unlike the Tukey order, the wRK order preserves definable complexity. This is because the map

\[ X \mapsto f^{-1}[X] \]

is continuous on \( 2^\omega \) for any \( f : \omega \to \omega \).

Thus, \( \ell_1 \) is not a wRK-maximal P-ideal since it is merely \( F_\sigma \).

wRK \( \implies \) Tukey

If \( f : A \to \omega \) is a wRK map witnessing \( I \leq_{wRK} J \), then

\[ X \mapsto f^{-1}[X] \]

is a Tukey map.
An analogue of Fremlin’s result

Beros–Larson
There is a wRK-maximal $\Sigma^1_1$ P-ideal.

Crucial component (S. Solecki)
Every $\Sigma^1_1$ P-ideal on $\omega$ is of the form

$$Exh(\phi) = \{ X \subseteq \omega : \lim_{n} \phi(X \setminus n) = 0 \}$$

for some lower semicontinuous submeasure $\phi : \mathcal{P}(\omega) \to \mathbb{R} \cup \{\infty\}$.

Example
If $\phi(X) = \sup \{|F|/(\max(F) + 1) : F \in [X]^{<\omega}\}$, then $Z_0 = Exh(\phi)$.

A helpful refinement
The lsc submeasure $\phi$ can be chosen with $\phi(F) \in \mathbb{Q}$ for each finite $F \subseteq \omega$. This makes it easy to code all lsc submeasures.
The idea behind the proof

Beros–Larson
There is a wRK-maximal $\Sigma^1_1$ P-ideal.

Proof idea
Choose a perfect almost disjoint family $\{A_\alpha : \alpha \in \omega^\omega\} \subseteq \mathcal{P}(\omega)$.

Each $\alpha$ will code an lsc submeasure via “building instructions”.

Define a “universal” lsc submeasure $\phi_{max}$ such that, for all $\phi$,

$$\text{Exh}(\phi) \leq_{\text{wRK}} \text{Exh}(\phi_{max})$$

is witnessed by a bijection $f_\alpha : A_\alpha \to \omega$ where $\alpha$ codes $\phi$.

Key property: $\phi(X) = \phi_{max}(f_\alpha^{-1}[X])$. 
Well-founded trees

Let \( I_\alpha \) be the set of \( X \subseteq 2^{<\omega} \) such that

\[
\text{rank}(X, \preceq) < \alpha
\]

If \( \alpha = \omega^\beta \) (i.e., \( \alpha \) is additively closed), then \( I_\alpha \) is an ideal. This follows from results of Ryan Causey. Remark: the sets \( X \) are not necessarily subtrees of \( 2^{<\omega} \).

Example

\( I_\omega \) is the ideal of finite rank sets of strings, i.e., the ideal generated by the \( \preceq \)-antichains.

Beros–Larson

If \( \alpha = \omega^{\beta+1} \), then \( I_\omega \leq_{\text{WRK}} I_\alpha \).
More Tukey results

Beros–Larson
The ideal $I_\omega$ is Tukey-maximal among partial orders of size continuum. (Because it contains a $c$-size strongly unbounded set.) Often (though not always), the Tukey order is “indifferent” to definable complexity. Nevertheless, here is an example of interplay between the Tukey order and definable complexity.

Beros–Larson
If $I$ is an $F_\sigma$ ideal with $I \leq_{Tukey} NWD$, then $I$ is countably generated.

This is of note because $NWD$ itself is not countably generated and is Tukey above non countably generated $G_{\delta_\sigma}$ ideals.

This gives, for instance, a simple proof that $\ell_1 \nless_{Tukey} NWD$. 