Normal numbers in the Hausdorff hierarchy

Konstantinos A. Beros

University of North Texas

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The Wadge hierarchy

Wadge reducibility
Polish spaces $X$ and $Y$. Subsets $A \subseteq X$ and $B \subseteq Y$. Define

$$A \leq_W B \iff \exists \text{cts } f : X \to Y \text{ s.t. } A = f^{-1}(B)$$

Wadge completeness
Pointclass $\Gamma$. Polish space $X$. Subset $B \in \Gamma(X)$ is $\Gamma$-complete iff $A \leq_W B$, for each $A \in \Gamma(2^\omega)$.

Example
$$\{ x \in 2^{\omega \times \omega} : (\forall n)(\forall k) \ x(n, k) = 0 \} \text{ is } \Pi^0_3\text{-complete.}$$

Remark
Wadge completeness is often used to show that a set is properly in a pointclass $\Gamma$. 
Normal numbers

$k$-normality
Let $T : [0, 1] \to [0, 1]$ be given by $T(x) = 2x \ (\text{mod } 1)$ (the unilateral shift on binary expansions). A real number $x \in [0, 1]$ is $k$-normal (to base 2) iff

$$\lim_{n \to \infty} \frac{1}{n} \left| \{ i < n : T^i(x) \in I \} \right| = 2^{-k},$$

for each of the intervals $I = [s \ 2^{-k}, (s + 1) \ 2^{-k})$, with $0 \leq s < 2^k$.

Let $N_k$ denote the set of $k$-normal numbers.

Note that $N_1 \supset N_2 \supset N_3 \supset \ldots$

A real number is normal (to base 2) iff it is $k$-normal, for every $k$.

Combinatorial formulation
A real number $x$ is $k$-normal iff every binary string of length $k$ occurs with limiting frequency $2^{-k}$ in the binary expansion of $x$. 
Ki-Linton (1994)

- Each $N_k$ is $\Pi^0_3$-complete.
- The set of normal numbers (to a fixed base) is $\Pi^0_3$-complete.

Becher-Heiber-Slaman (2014)

The set of numbers normal to every base is $\Pi^0_3$-complete.

Becher-Slaman (2014)

The set of numbers normal to at least one base is $\Sigma^0_4$-complete.
New results

The difference hierarchy

- Provides an $\omega_1$-length hierarchy within each $\Delta_\alpha^0$.
- For instance, $D_2(\Pi_3^0)$ is the class of differences of $\Pi_3^0$ sets.
- Also, $D_\omega(\Pi_3^0)$ is the class of sets of the form

$$
(H_1 \setminus H_2) \cup (H_3 \setminus H_4) \cup \ldots,
$$

where $H_1 \supseteq H_2 \supseteq \ldots$ are $\Pi_3^0$ sets.
- Wadge-completeness results in the difference hierarchy are uncommon.

K. Beros

- The set $N_1 \setminus N_2$ is $D_2(\Pi_3^0)$-complete.
- The set $(N_1 \setminus N_2) \cup (N_3 \setminus N_4) \cup \ldots$ is $D_\omega(\Pi_3^0)$-complete.
Theorem
The set $N_1 \setminus N_2$ is $D_2(\Pi^0_3)$-complete.

General approach
Given $\Pi^0_3$ sets $L \subseteq H \subseteq 2^\omega$, define a continuous map $f : 2^\omega \to [0,1]$ such that

$$H = f^{-1}(N_1) \text{ and } L = f^{-1}(N_2).$$

Basic building blocks
Observe that $0.011001100110\ldots$ is 2-normal

$\alpha_n = (0110)^n \sim 01$

$\beta_n = (0110)^n \sim 0$

Higher indexed $\alpha_n$ and $\beta_n$ are closer to being the binary expansions of a 2-normal number.
A permitting condition

Let \langle \cdot, \cdot \rangle : \omega^2 \to \omega be a bijection which is increasing in the second coordinate.

Say \( L = \bigcap_n L_n \), with each \( L_n = \bigcup_p L_{n,p} \) a \( \Sigma^0_2 \) set. Given \( x \in 2^{\omega} \) and \( t = \langle n, p \rangle \), say that \( x \) “appears to be in \( L \) at stage \( t \)” if

\[
\text{dist}(x, L_{n,p}) \leq 1/\langle n, p \rangle
\]

and, for all \( p' < p \),

\[
\text{dist}(x, L_{n,p'}) \leq 1/\langle n, p - 1 \rangle \implies \text{dist}(x, L_{n,p'}) \leq 1/\langle n, p \rangle.
\]

Similarly, one may describe \( x \) as “appearing to be in \( H \) at stage \( t \)”.

Lemma

\( x \in L_n \) iff \( x \) appears to be in \( L \) at stage \( \langle n, p \rangle \), for all but finitely many \( p \). And analogously for \( H \).
Recall: $\alpha_n = (0110)^n \downarrow 01$

Given $x \in 2^\omega$, define binary strings $\sigma_0 \prec \sigma_1 \prec \ldots$ by induction. If $x$ appears to be in $L$ at stage $t = \langle n, p \rangle$, let

$$\sigma' = \sigma_{t-1} \downarrow \alpha_t$$

and, if $x$ does not appear to be in $L$ at stage $t = \langle n, p \rangle$, let

$$\sigma' = \sigma_{t-1} \downarrow (\alpha_n)^k,$$

where $k$ is large enough to skew the frequency of the string 01 in $\sigma'$ away from $\frac{1}{4}$.

Notice that this does not skew the frequency of 0 and 1 in $\sigma'$. 
Recall: $\beta_n = (0110)^n \downarrow 0$

If $x$ appears to be in $H$ at stage $t = \langle n, p \rangle$, let

$$\sigma_t = \sigma' \setminus \beta_t$$

and, if $x$ does not appear to be in $H$ at stage $t$, let

$$\sigma_t = \sigma' \setminus (\beta_n)^k,$$

where $k$ is large enough to skew the frequency of 0 away from $\frac{1}{2}$.

Let $f(x) = 0. \lim \sigma_t$. 
Suppose $x \in L$

$\implies (\forall n)(\forall \infty p) \ x$ appears to be in both $L$ and $H$ at stage $\langle n, p \rangle$

$\implies$ each $\alpha_n$ and $\beta_n$ appears only finitely many times in $\lim \sigma_t$

$\implies f(x) \in N_2$

Suppose $x \notin L$

$\implies (\exists n)(\exists \infty p) \ x$ does not appear to be in $L$ at stage $\langle n, p \rangle$

$\implies \exists n$ there are infinitely many long blocks of $\alpha_n$, which skew the frequency of 01 in $\lim \sigma_t$

$\implies f(x) \notin N_2$
Note

\[ x \in L \text{ vs. } x \notin L \text{ does not affect the frequency of 0 and 1 in } \lim \sigma_t, \]
i.e., it does not affect \( f(x) \in N_1 \).

Thus, the same argument as above shows that

\[ x \in H \iff f(x) \in N_1. \]

This completes the proof.