On a Conjecture about Trees in Graphs with Large Girth

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The girth of a graph $G$ is the length of a shortest cycle in $G$. Dobson (1994, Ph.D. dissertation, Louisiana State University, Baton Rouge, LA) conjectured that every graph $G$ with girth at least $2t+1$ and minimum degree at least $kt$ contains every tree $T$ with $k$ edges whose maximum degree does not exceed the minimum degree of $G$. The conjecture has been proved for $t \leq 3$. In this paper, we prove Dobson’s conjecture.

Key Words: trees; girth.

1. INTRODUCTION

In 1964, Erdős and Sós made the following well-known conjecture.

Conjecture 1. Every graph of order $n$ with more than $n(k-1)/2$ edges contains every tree with $k$ edges as a subgraph.

The conjecture has been verified only for a few special cases. In 1996, Brandt and Dobson [1] proved the conjecture for graphs with girth at least 5.

Theorem A. Every graph of order $n$ and girth at least 5 with more than $n(k-1)/2$ edges contains every tree with $k$ edges as a subgraph.

This follows from their slightly stronger result

Theorem B. Let $G$ be a graph with girth at least 5 and $T$ be a tree with $k$ edges. If $\delta(G) \geq k/2$ and $\Delta(G) \geq \Delta(T)$, then $G$ contains $T$ as a subgraph.

By replacing $\Delta(G) \geq \Delta(T)$ with the stronger condition $\delta(G) \geq \Delta(T)$, Dobson [2] made the following general conjecture.

Conjecture 2. Let $G$ be a graph with girth at least $2t+1$ and $T$ be a tree with $k$ edges. If $\delta(G) \geq k/t$ and $\delta(G) \geq \Delta(T)$, then $G$ contains $T$ as a subgraph.
Note that if the condition \( \delta(G) \geq \Delta(T) \) is replaced with \( \Delta(G) \geq \Delta(T) \), then the conjecture does not hold for \( t \geq 3 \). For instance, for \( t = 3, k \geq 9 \), let \( T \) be the double star with \( k \) edges in which the two non-leaf vertices have degree \( \lceil \frac{k+1}{2} \rceil \) and \( \lceil \frac{k-1}{2} \rceil \), respectively, and define \( G \) as follows. Let \( H \) be a \( \lceil k/3 \rceil \)-regular graph with girth 7. Form \( G \) from a disjoint union of at least \( \lceil \frac{k-1}{2} \rceil \) copies of \( H \) by adding a vertex, adjacent to exactly one vertex in each copy of \( H \). It is easy to check that \( G \) has girth 7, maximum degree at least \( \lceil \frac{k+1}{2} \rceil \), and minimum degree at least \( \lceil k/3 \rceil \). However, \( T \) cannot be embedded in \( G \), since \( G \) has only one vertex whose degree is at least \( \lceil \frac{k+1}{2} \rceil \). This example can clearly be generalized for all \( t \geq 3 \).

Conjecture 2 is known to be true for \( t \geq 3 \). The fact that it holds for \( t = 1 \) is well known (see [7]). The case \( t = 2 \) is implied by Theorem B. The case \( t = 3 \) was proved by Sacle and Woźniak [5].

In this paper, we prove the conjecture. We state the result in the following equivalent form.

**Theorem 1.** Let \( G \) be a graph with girth at least \( 2t + 1 \) and \( T \) be a tree with at most \( k \) edges. If \( \delta(G) \geq k/t \) and \( \delta(G) \geq \Delta(T) \), then \( G \) contains \( T \) as a subgraph.

2. **Preliminaries**

We first introduce some terminology. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [7].

Given graphs \( G \) and \( H \), an embedding of \( H \) in \( G \) is an injection \( f: V(H) \to V(G) \) such that \( uv \in E(H) \) implies \( f(u)f(v) \in E(G) \). Let \( T \) be a tree; a leaf in \( T \) is a vertex with degree 1. Two leaves with the same neighbor in \( T \) are siblings. The derived tree of \( T \), denoted by \( D(T) \), is the subtree of \( T \) obtained by deleting all the leaves of \( T \). For a subtree \( T' \) of \( D(T) \), \( L(T') \) denotes the subtree of \( T \) containing \( T' \) and all the leaves in \( T \) that are adjacent to \( V(T') \). A penultimate vertex in \( T \) is a leaf in \( D(T) \). For a positive integer \( m \), \( \lfloor m \rfloor \) denotes the set of integers 1, \ldots, \( m \). A component in a graph is nontrivial if it contains at least two vertices. If \( P \) is a path and \( x, y \) are two vertices on \( P \), then \( P[x, y] \) denotes the portion of \( P \) between \( x \) and \( y \).

Next, we develop some useful tools in the following lemmas. In our proofs of the lemmas and Theorem 1, we will implicitly use the fact that a closed walk containing an edge that is used only once necessarily contains a cycle. For the rest of the paper, we assume that \( t \geq 2 \). We begin with an easy observation. We omit the proof since it is straightforward.

**Lemma 1.** Let \( G \) be a graph with girth at least \( 2t + 1 \) and \( T \) be a tree with diameter at most \( 2t \). If \( \delta(G) \geq \Delta(T) \), then \( G \) contains \( T \) as a subgraph.
Furthermore, if \(x, y\) are two adjacent vertices in \(G\) and \(u, v\) are two adjacent vertices in \(T\), then \(T\) can be embedded in \(G\) in such a way that \(u, v\) are mapped to \(x, y\) respectively.

**Lemma 2.** Let \(G\) be a connected graph of order \(n\) and \(S\) be a subset of \(V(G)\) such that every pair in \(S\) has distance at least \(2t-1\) in \(G\). Then \(|S| \leq \max\{\lceil t/2 \rceil, 1\} \).

**Proof.** We may assume that \(S\) contains at least two vertices. Suppose \(S = \{v_1, \ldots, v_m\}\), where \(m \geq 2\). For each \(i \in [m]\), let \(N_i = \{u \in V(G) : \text{dist}_G(u, v_i) \leq t - 1\}\). Since \(G\) is connected and a shortest path connecting any pair \(v_i, v_j\) has length at least \(2t-1\), we have \(|N_i| \geq 1 + (t-1) = t\), for each \(i \in [m]\), and \(N_i \cap N_j = \emptyset\) for \(i \neq j\). Hence \(n \geq |\bigcup_{i=1}^{m} N_i| = \sum_{i=1}^{m} |N_i| \geq mt\), and therefore \(m \leq \lceil t/2 \rceil\).

Lemma 2 is best possible for trees. The tree \(T\) obtained by identifying each vertex on a \(P_m\) with an endpoint of a \(P_i\) has \(mt\) vertices and its leaves form a subset of size \(m\) with pairwise distance at least \(2t-1\) in \(T\).

Fixing \(k, t, \) and a graph \(G\) with a girth of at least \(2t+1\) and a minimum degree of at least \(k/t\), a minimal nonembeddable tree is a tree \(T\) with at most \(k\) edges and maximum degree at most \(\delta(G)\), such that \(T\) cannot be embedded in \(G\) but every proper subtree of \(T\) can be embedded in \(G\). The next two lemmas reveal some properties that such a minimal nonembeddable tree would have.

**Lemma 3.** Let \(T\) be a minimal nonembeddable tree for some fixed \(k, t\), \(G\). Then \(T\) contains at most \(t-1\) penultimate vertices.

**Proof.** Suppose otherwise that \(T\) contains at least \(t\) penultimate vertices. Let \(w\) be a penultimate vertex of \(T\) with the least number of adjacent leaves. Let \(v_1, \ldots, v_l\) be its adjacent leaves. Let \(w_1, \ldots, w_{l-1}\) denote \(t-1\) other penultimate vertices in \(T\), each adjacent to at least \(l\) leaves. By our assumption, there exists an embedding \(f\) of \(T' = T - \{v_1, \ldots, v_l\}\) in \(G\). Let \(T' = T - w\). Then \(T'\) is a subtree of \(T\) with at most \((k+1) - (l+1) = k - l\) vertices. Let \(S\) denote the set of neighbors (in \(G\) of \(f(w)\) in \(V(f(T'))\)). Since \(G\) has girth at least \(2t+1\), the distance in \(f(T')\) between any pair of vertices of \(S\) is at least \(2t-1\), which is at least \(3\) for \(t \geq 2\). In particular, no two sibling leaves in \(f(T')\) can belong to \(S\) at the same time. Hence at most one of the leaves adjacent to \(f(w_i)\) can be a member of \(S\), for \(i \in [t-1]\). This means that for each \(i \in [t-1]\) we can delete \(l-1\) leaves adjacent to \(f(w_i)\) in \(f(T')\) without deleting a member of \(S\). Denote the subtree of \(f(T')\) obtained in this way by \(T^*\). \(T^*\) has \(m(f(T')) - (t - 1)(l-1) \leq (k-l) - (t-1)(l-1) = k - ll + (t-1)\) vertices. The distance between every pair in \(S\) remains the same in \(T^*\) as in \(f(T')\). We may further assume that \(n(T^*) \geq t,\)
since otherwise $T^*$ has diameter at most $t - 2$ and $T$ would have diameter at most $(t - 2) + 2 = t$, contradicting Lemma 1. By Lemma 2, we have $|S| \leq \max(\lfloor \frac{m(T)}{2} \rfloor, 1) = \lfloor \frac{k - d + (t - 1)}{2} \rfloor = \lfloor \frac{k}{2} \rfloor - 1$. Hence $f(w)$ has at most $\lfloor \frac{k}{2} \rfloor - 1$ neighbors in $V(f(T^*))$. Since $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$, $f(w)$ has at least $l$ neighbors outside $V(f(T^*))$. In that case, we can extend $f$ to an embedding of $T$ in $G$ by embedding $v_1, ..., v_l$ in the neighborhood of $f(w)$ outside $V(f(T^*))$, contradicting our assumption. 

**Lemma 4.** Let $T$ be a minimal nonembeddable tree for some fixed $k$, $t$, $G$. Let $x$ be a vertex in $T$ such that every nontrivial component in $T - x$ has at least $t$ vertices. Then $x$ is adjacent to no leaves of $T$ or to at least two leaves of $T$. In particular, each penultimate vertex in $T$ is adjacent to at least two leaves.

**Proof.** To prove the first part of the claim, suppose that $x$ is adjacent to exactly one leaf $v$ in $T$. Let $T' = T - v$. By our assumption, there exists an embedding $f$ of $T'$ in $G$. Let $T_1, ..., T_m$ denote the components in $T' - x$, with $n_1, ..., n_m$ vertices respectively. Each $T_i$ is nontrivial, hence $n_i \geq t$. Let $S_i$ denote the set of neighbors of $f(x)$ in $V(f(T_i))$. Since $G$ has girth at least $2t + 1$, every pair in $S_i$ has a distance of at least $2t - 1$ in $f(T_i)$. By Lemma 2, $|S_i| \leq \max\left(\lfloor \frac{m(T)}{2} \rfloor, 1\right) = \lfloor \frac{k}{2} \rfloor - 1$. Hence $f(x)$ has at most $\sum_{i=1}^m (n_i/t) = \lfloor \frac{m(T)}{2} \rfloor = \lfloor \frac{k}{2} \rfloor - 1$ neighbors in $V(f(T' - x))$. Since $\delta(G) \geq \lfloor \frac{k}{2} \rfloor$, $f(x)$ has at least one neighbor outside $V(f(T' - x))$, in which case we can extend $f$ to an embedding of $T$ in $G$ by mapping $v$ to a neighbor of $f(x)$ outside $V(f(T' - x))$, contradicting our assumption. This proves the first part of the claim.

Now, let $x$ be a penultimate vertex of $T$. By Lemma 1, we may assume that $T$ has diameter at least $2t + 1$, in which case $T - x$ consists of isolated vertices and a component of diameter at least $2t - 1$; such a component has at least $t$ vertices. By the first part of the claim, $x$ is adjacent to either none or to at least two leaves in $T$. Since $x$ is adjacent to at least one leaf, $x$ must be adjacent to at least two leaves. 

We omit the proof of the next lemma since it is straightforward.

**Lemma 5.** A tree with $l$ leaves has at most $l - 2$ vertices of degree at least 3.

### 3. PROOF OF THEOREM 1

**Proof of Theorem 1.** Fixing $k$, $t$, and a graph $G$ with girth at least $2t + 1$ and minimum degree at least $k/t$, we prove that a minimal nonembeddable
tree does not exist. This will prove the theorem. Suppose otherwise that there exists a minimal nonembeddable tree $T$ with at most $k$ edges and maximum degree at most $\delta(G)$. Without loss of generality, we may assume that $T$ has $k$ edges. By Lemma 3 and Lemma 4, $T$ contains at most $t - 1$ penultimate vertices, each adjacent to at least two leaves. Hence the derived tree $D(T)$ contains at most $t - 1$ leaves. If $D(T)$ does not contain any vertex of degree 2, then by Lemma 5 $D(T)$ contains at most $t - 3$ nonleaf vertices, in which case $D(T)$ has diameter at most $t - 2$ and $T$ has diameter at most $t - 2$, contradicting Lemma 1. Hence we may assume that $D(T)$ contains at least one vertex of degree 2.

Given a vertex $x$ in $D(T)$ and a component $A$ in $D(T) - x$, we define the depth of $A$ from $x$, denoted by $\text{dep}(x, A)$, as $\max\{\text{dist}_{D(T)}(x, u) : u \in V(A)\}$ and define the depth of $x$, denoted by $\text{dep}(x)$, as $\min\{\text{dep}(x, A) : A$ is a component in $D(T) - x\}$.

Given positive integers $a, b$ with $a \geq 2b$, let $R(a, b)$ denote the tree with $2b$ leaves obtained from the path $v_1 \cdots v_a$ by adding a leaf adjacent to each of $v_2, \ldots, v_b, v_{a-(b-1)}, \ldots, v_{a-1}$.

CLAIM 1. Let $m = \min\{\text{dep}(x) : x$ is a vertex of degree 2 in $D(T)\}$. Then $m \leq (t - 1)/2$. Furthermore, if $m = (t - 1)/2$, then $D(T) = R(r, m)$ for some $r \geq 2m$, and each vertex of degree 2 in $D(T)$ is adjacent to either none or at least two leaves in $T$.

Proof. Let $x$ be any vertex of degree 2 in $D(T)$ and $A_1, A_2$ be the two components in $D(T) - x$. Let $u$ be a vertex in $A_1$ at maximum distance from $x$ in $D(T)$ and $v$ be a vertex in $A_2$ at maximum distance from $x$ in $D(T)$. Let $P_1 = u_0u_1 \cdots u_l$ and $P_2 = v_0v_1 \cdots v_h$ denote the $u, x$-path and the $v, x$-path in $D(T)$, respectively, where $u_0 = u$, $v_0 = v$, and $u_l = x = v_h$.

Let $p \leq l$ denote the smallest subscript such that $u_p$ has degree 2 in $D(T)$ and $q$ denote the smallest subscript such that $v_q$ has degree 2 in $D(T)$. Let $A_p$ denote the component in $D(T) - u_p$ containing $u$ and $A_q$ denote the component in $D(T) - v_q$ containing $v$. It is clear, by our choice of $u, v$, that $A_p$ has depth $p$ from $u$, and $A_q$ has depth $q$ from $v$. Hence, $p \geq m, q \geq m$. For $i \in \{p - 1\}$, $u_i$ has degree at least 3 in $D(T)$ and hence leads to at least one leaf in $D(T)$; such a leaf clearly lies in $A_p$. Since $u_0 = u$ is also a leaf in $D(T)$, $A_p$ contains at least $p \geq m$ leaves of $D(T)$ and at least $(p - 1) + p = 2p - 1 \geq 2m - 1$ vertices. If $A_p$ contains exactly $m$ leaves, then $p = m$ and $A_p$ cannot contain any vertex having degree 2 in $D(T)$, since such a vertex would have depth less than $p = m$. In that case, it is clear that $A_p$ contains exactly the path $u_0u_1 \cdots u_{m-1}$ plus one leaf adjacent to each of $u_1, \ldots, u_{m-1}$.

Similarly, $A_q$ contains at least $q \geq m$ leaves of $D(T)$ and at least $2q - 1 \geq 2m - 1$ vertices. If $A_q$ contains exactly $m$ leaves, then $q = m$ and $A_q$ contains exactly the path $v_0v_1 \cdots v_{m-1}$, plus a leaf adjacent to each of $v_1, \ldots, v_{m-1}$.
Hence, all together $D(T)$ contains at least $p + q \geq 2m$ leaves. Since $D(T)$ contains at most $t - 1$ leaves, we have $2m \leq t - 1$ or $m \leq (t - 1)/2$. If $m = (t - 1)/2$, then $D(T)$ contains exactly $2m$ leaves, which happens only if $A_1, A_2$ each contains exactly $m$ leaves of $D(T)$ and all the vertices in $D(T) - A_1 - A_2$ have degree 2 in $D(T)$. In that case, we have $D(T) = R(l + h + 1, m)$.

The two nontrivial components in $T - x$ are exactly $L(A_1)$ and $L(A_2)$. $L(A_1)$ contains at least $2m - 1$ vertices in $A_p$ and at least two leaves in $T$ adjacent to $u$, hence $L(A_1)$ contains at least $2m + 1$ vertices. Similarly, $L(A_2)$ contains at least $2m + 1$ vertices. If $m = (t - 1)/2$, then $2m + 1 = t$. By Lemma 4, $x$ is adjacent to either no leaves of $T$ or at least two leaves of $T$.

**Remark 1.** Let $x$ be an arbitrary vertex of degree 2 in $D(T)$. By the definition of $m$, we have $\text{dep}(x) \geq m$. Hence, if $A$ is a component in $D(T) - x$ then $A$ has depth at least $m$ from $x$ by the definition of $\text{dep}(x)$. (See the corresponding definitions regarding depth in the second paragraph prior to Claim 1.)

Now, let $w$ be a vertex of degree 2 in $D(T)$ with depth $m$, where $m$ is defined as in Claim 1, and let $A_1, A_2$ denote the two components of $D(T) - w$, where $A_1$ has depth $m$ from $w$. Let $w'$ denote the unique neighbor of $w$ in $A_1$, and let $w''$ denote the unique neighbor of $w$ in $A_2$. We have $\text{dist}_{D(T)}(w', x) \leq m - 1$, $\forall x \in V(A_1)$.

The two components of $D(T) - w w'$ are $A_1$ and $A_2 \cup w w''$ (adding an edge always includes adding its endpoints). So, the two components of $T - w w'$ are $L(A_1)$ and $L(A_2 \cup w w'')$. Let $T_1 = L(A_1) \cup w w'$, and let $T_2 = L(A_2 \cup w w'') \cup w w''$. $T_1$ and $T_2$ are proper subtrees of $T$ with $T_1 \cap T_2 = T$ and $T_1 \cap T_2 = w w'$. Furthermore, since $\text{dist}_{D(T)}(w', x) \leq m - 1$, $\forall x \in V(A_1)$, we have $\text{dist}_{T_1}(w', y) \leq m$, $\forall y \in V(T_1)$. In particular, this implies that $T_1$ has diameter at most $2m$.

Each $T_i$ is a proper subtree of $T$, hence can be embedded in $G$. We show that we can embed $T_1$ and $T_2$ in $G$ in an appropriate way, such that the two embeddings together form an embedding of $T$ in $G$, which will contradict our assumption about $T$ and complete the proof.

Let $f$ be an embedding of $T_2$ in $G$. We define several subsets of $N_G(f(w))$ which count different types of neighbors of $f(w)$. Let $T'' = L(A_2)$ (hence $T''$ is obtained from $T_2$ by deleting $w$ and its adjacent leaves in $T_2$), and let $H = f(T'')$. Let

$$N_1 = N_G(f(w)) \cap V(H),$$

$$N_2 = \{ u \in N_G(f(w)) - V(H) : \text{dist}_{G - f(w)}(u, V(H)) \leq m \},$$

(1)
and
\[ N_3 = \{ u \in N_G(f(w)) : V(H) : \text{dist}_{G-f(w)}(u, V(H)) > m \}. \]

Clearly, \( N_G(f(w)) = N_1 \cup N_2 \cup N_3 \) and \( N_1, N_2, N_3 \) are pairwise disjoint.

**Claim 2.** If \( N_3 \neq \emptyset \), then \( T \) can be embedded in \( G \).

**Proof.** Suppose \( N_3 \) is nonempty. Note that \( f \) maps the leaves adjacent to \( w \) in \( T_2 \) (including \( w' \)) into \( N_2 \cup N_3 \). Rearranging the images of those leaves within \( N_2 \cup N_3 \) does not change the rest of the embedding or the sets \( N_1, N_2, N_3 \). Hence we may assume without loss of generality that \( f(w') \in N_3 \). (In other words, we may modify \( f \) if necessary so that \( w' \) is mapped to a vertex in \( N_3 \).

Since \( T_1 \) has diameter at most \( 2m \leq 2t \), by Lemma 1 there exists an embedding \( \phi \) of \( T_1 \) in \( G \) such that \( \phi(w') = f(w') \) and \( \phi(w) = f(w) \). We show that \( V(\phi(T_1)) \cap V(f(T_2)) = \{ f(w'), f(w) \} \), in which case \( \phi(T_1) \cup f(T_2) \) forms a copy of \( T \) in \( G \).

Suppose there exists a vertex \( z \in V(\phi(T_1)) \cap V(f(T_2)) \setminus \{ f(w'), f(w) \} \). Let \( x_1 = \phi^{-1}(z) \in V(T_1) \setminus \{ w', w \} \). By our earlier discussion, we have \( \text{dist}_{T_1}(w', x_1) \leq m \). Note that the unique \( w' \)-\( x_1 \)-path in \( T_1 \) does not use \( w \). Thus the image (under \( \phi \)) of that path is a \( \phi(w') \)-\( z \)-path of length at most \( m \) that avoids \( \phi(w) \). Hence, we have \( \text{dist}_{G-f(w'), \phi(w')}(\phi(w'), z) \leq m \) or, equivalently, \( \text{dist}_{G-f(w'), \phi}(f(w'), z) \leq m \).

Let \( x_2 = f^{-1}(z) \in V(T_2) \setminus \{ w', w \} \). If \( x_2 \) is a leaf in \( T_2 \) adjacent to \( w \), then the cycle formed by the edges \( zf(w), f(w) f(w') \) and a shortest \( f(w') \)-\( z \)-path in \( G-f(w) \) has length at most \( m + 2 \leq 2t \), contradicting the girth requirement. Hence \( x_2 \in V(T') \), and therefore \( z \in V(H) \). Now, we have \( \text{dist}_{G-f(w'), \phi}(f(w'), V(H)) \leq \text{dist}_{G-f(w'), \phi}(f(w'), z) \leq m \), contradicting our assumption that \( f(w') \in N_3 \).

It remains to show that \( N_3 \neq \emptyset \). Since \( |N_G(f(w))| \geq \lceil \frac{n}{2} \rceil \) and \( N_1, N_2, N_3 \) are pairwise disjoint, it suffices to show that \( |N_1 \cup N_2| \leq \lceil \frac{n}{2} \rceil - 1 \). Suppose \( N_2 = \{ a_1, ..., a_p \} \). By definition, for each \( i \in [p] \), there exists some \( a_i^* \in V(H) \) such that \( \text{dist}_{G-f(w), \phi}(a_i, a_i^*) \leq m \). Let \( N_2^* = \{ a_1^*, ..., a_p^* \} \). We claim that \( a_1^*, ..., a_p^* \) are all distinct and \( N_2^* \cap N_1 = \emptyset \). Suppose that \( a_i^* = a_j^* \) for some \( i \neq j \), then we would obtain a cycle of length at most \( 2m + 2 \leq 2t \) from the union of a shortest \( a_i \)-\( a_j \)-path in \( G-f(w) \), a shortest \( a_i \)-\( a_j \)-path in \( G-f(w) \), and the edges \( a_i f(w) \) and \( a_j f(w) \), a contradiction. Hence the \( a_i^* \)'s are all distinct. If \( a_i^* \in N_1 \) for some \( i \), then again we can obtain a cycle of length at most \( 2m + 2 \leq 2t \) from the union of a shortest \( a_i \)-\( a_i^* \)-path in \( G-f(w) \) and the edges \( a_i f(w) \), \( a_i^* f(w) \), a contradiction. Hence \( N_2^* \cap N_1 = \emptyset \).
Let $S = N_1 \cup N_2^*$. Then $S$ is a subset of $V(H)$, and the above discussion shows that $|S| = |N_1 \cup N_2^*| = |N_1| + |N_2^*| = |N_1| + |N_2| = |N_1 \cup N_2|$. So it suffices to show that $|S| \leq \lceil \frac{k}{2} \rceil - 1$.

**Claim 3.** $|S| \leq \lceil \frac{k}{2} \rceil - 1$.

**Proof.** By Lemma 1, we may assume that $T$ has a diameter of at least $2t + 1$, in which case $T^*$ has a diameter of at least $2t + 1 - (m + 2) \geq t$ and hence contains at least $t$ vertices. Since $T^* = L(A_2)$ omits vertices in $L(A_1)$ and $w$, and $L(A_1)$ contains at least $2m + 1$ vertices by the proof of Claim 1, $T^*$ contains at most $(k + 1) - (2m + 1) - 1 = (k - 1) - 2m$ vertices. Hence $H$ contains at most $(k - 1) - 2m$ vertices and at least $t$ vertices. (Recall that $H = f(T^*)$.)

Using girth type arguments as before, we have

$$\text{dist}_H(u, v) > 2t - 2 \quad \text{ if } u, v \in N_1, \quad u \neq v,$$

(2)

$$\text{dist}_H(u, v) > 2t - 2(m + 1) \quad \text{ if } u, v \in N_2^*, \quad u \neq v,$$

and

$$\text{dist}_H(u, v) > 2t - m - 2 \quad \text{ if } u \in N_1, \quad v \in N_2^*.$$

Consider the case $u \in N_1, \quad v \in N_2^*$, for instance. If $\text{dist}_H(u, v) \leq 2t - m - 2$, then the union of a shortest $u, v$-path in $H$, a shortest $v, f(w')$-path in $G - f(w)$, and the edges $f(w') f(w), u f(w)$ contains a cycle of length at most $(2t - m - 2) + m + 2 = 2t$, a contradiction.

For each vertex $x \in S = N_1 \cup N_2^*$, define $B(x)$ as follows.

$$B(x) = \{ y \in V(H) : \text{dist}_H(x, y) \leq t - 1 \} \quad \text{ if } x \in N_1,$$

(3)

and

$$B(x) = \{ y \in V(H) : \text{dist}_H(x, y) \leq t - (m + 1) \} \quad \text{ if } x \in N_2^*.$$

By (2) and (3), we have $B(x) \cap B(x') = \emptyset$ for distinct vertices $x, x' \in S$.

Since $H$ is connected and has at least $t$ vertices, it is immediate from the definition that $|B(x)| \geq \min\{1 + (t - 1), m(H)\} = t$ if $x \in N_1$ and that $|B(x)| \geq \min\{1 + [t - (m + 1)], m(H)\} = t - m$ if $x \in N_2^*$.

We show that in fact for $x \in N_2^*$ we also have $|B(x)| \geq t$, with at most two exceptions. It will then follow that $(k - 1) - 2m \geq m(H) \geq |\bigcup_{x \in S} B(x)| = \sum_{x \in S} |B(x)| \geq (|S| - 2) t + 2(t - m)$, from which we will get $|S| \leq \lceil \frac{k}{2} \rceil - 1$. 

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We consider two cases.

Case 1. \( m = (t - 1)/2 \).

In this case, \( t - (m + 1) = m \), and we know from the proof of Claim 1 that \( A_2 \) consists of a path \( v_0v_1 \cdots v_h \), where \( v_h = w^j \) is the unique neighbor of \( w \) in \( A_2 \), and a leaf \( u_i \) adjacent to \( v_i \) for each \( i = 1, \ldots, m - 1 \). Furthermore, each of \( v_m \) is a vertex of degree 2 in \( D(T) \) and is adjacent to either none or at least two leaves in \( T \).

Let \( F \) denote the subgraph of \( A_2 \) induced by \( v_0, v_1, \ldots, v_{m-1} \) and \( u_1, \ldots, u_{m-1} \). Let \( F^* = L(F) \). It is easy to check that \( F^* \) (and hence \( f(F^*) \)) cannot contain three vertices with pairwise distance at least \( m + 2 \). Because every pair in \( N_2^* \) has distance (in \( H \)) at least \( 2t - 2(m + 1) + 1 \geq m + 2 \), we have \( |N_2^* \cap V(f(F^*))| \leq 2 \). So it suffices to show that \( |B(x)| \geq t \) for \( x \in N_2^* - V(f(F^*)) \). Such a vertex \( x \) is either \( f(v_i) \) or a leaf in \( H \) adjacent to \( f(v_i) \), for some \( j \geq m \). Since \( \text{dist}_H(f(v_i), f(v_k)) \geq \text{dist}_H(x, f(v_k)) - 1 \geq 2t - m - 2 > m \) (note that \( x \in N_2^* \) and \( f(v_k) \in N_1 \)), we also have \( j < h - m \).

If \( x = f(v_i) \), where \( m < j < h - m \), then \( f(v_{i-m}), \ldots, f(v_i), \ldots, f(v_{i+m}) \) all belong to \( B(x) \), since they have distance at most \( m = t - (m + 1) \) from \( x \) in \( H \). Hence \( |B(x)| \geq 2m + 1 = t \). If \( x \) is a leaf adjacent to \( f(v_i) \), then by our earlier discussion \( x \) has a sibling leaf \( x' \) adjacent to \( f(v_i) \). Now \( x, x', f(v_{i-m}), \ldots, f(v_{i+m}) \) are \( 2m + 1 = t \) vertices belonging to \( B(x) \).

Case 2. \( m < (t - 1)/2 \).

In this case, we have \( t - (m + 1) \geq m + 1 \). We prove that \( |B(x)| \geq t \) for all \( x \in N_2^* \). Recall that \( w^j \) is the unique neighbor of \( w \) in \( T' \). Since \( f(w^j) \in N_1 \) and \( x \in N_2^* \), by (2) we have \( \text{dist}_H(x, f(w^j)) \geq 2t - m - 1 > t - (m + 1) \). Let \( P = x_0x_1 \cdots x_{t-(m+1)} \), denote the initial portion of length \( t - (m + 1) \) on the unique \( x, f(w^j) \)-path in \( H \), where \( x_0 = x \). The \( t - m \) vertices on \( P \) clearly belong to \( B(x) \). Hence it suffices to show that \( B(x) \) contains at least \( m \) vertices.

Since \( x_1, \ldots, x_{t-(m+1)} \) have degree at least 2 in \( H \), \( f^{-1}(x_1), \ldots, f^{-1}(x_{t-(m+1)}) \) belong to \( D(T) \). Let \( I = \{ i \in [t - (m + 1)] : f^{-1}(x_i) \text{ is a vertex of degree 2 in } D(T) \} \).

Let \( j \in [t - (m + 1)] - I \), then \( f^{-1}(x_j) \) is either a leaf of \( D(T) \) (which is possible only if \( j = 1 \)) and \( f^{-1}(x_0) \) is a leaf of \( T \) or a vertex of degree at least three in \( D(T) \). In the former case, \( f^{-1}(x_j) \) is adjacent to at least two leaves of \( T \) (Lemma 4). Hence \( x_j \) is adjacent to at least one leaf \( z_j \) of \( H \) which is different from \( x_0 \); \( z_1 \) is a neighbor of \( x_1 \) in \( H \) not on \( P \). In the latter case, \( f^{-1}(x_j) \) has a neighbor \( v_j \) in \( D(T) \) which is not on \( f^{-1}(P) \). If \( v_j \) is not a leaf of \( D(T) \), then it has a neighbor \( v_j \) in \( D(T) \) not on \( f^{-1}(P) \). If \( v_j \) is a leaf of \( D(T) \), then it is adjacent to some leaf \( v_i \) of \( T \); clearly, \( v_j \) is not on \( f^{-1}(P) \). Let \( z_jf(v_j) \), and \( z_jf(v_j) \). We have \( z_j, z_j \in V(H) - V(P) \), \( \text{dist}_H(z_j, x) = \text{dist}_H(y, x_0) + 1 = j + 1 \), and \( \text{dist}_H(z_j, x) = \text{dist}_H(y, x_0) + 2 = j + 2 \).
Let \( b = \min \{ i : i \in I \} \). If \( b \geq m + 1 \) or \( b \) doesn't exist (i.e., if \( I = \emptyset \)), then \( [m] \cap I = \emptyset \). By our discussion above, for each \( j \in [m] \), \( x_j \) has a neighbor \( z_j \) in \( H \) not on \( P \) which is at a distance \( j + 1 \leq m + 1 \leq t - (m + 1) \) from \( x \). These \( z_j \)'s are clearly distinct from each other. This yields \( |B(x) - P| \geq m \) and we are done. We henceforth assume that \( b \leq m \).

By our definition of \( h, f^{-1}(x_h) \) is a vertex of degree of 2 in \( D(T) \). Let \( A_b \) denote the component in \( D(T) - f^{-1}(x_h) \) that doesn't contain \( w \). Then \( A_b \) is contained in \( A_b \subseteq T^* \) and does not contain \( f^{-1}(x_{b+1}), f^{-1}(x_{b-(m+1)}) \).

By Remark 1 (immediately after Claim 1), \( A_b \) has a depth at least \( m \) from \( f^{-1}(x_h) \). Hence there exists a path in \( A_b \cup f^{-1}(x_h) \) of length \( m \) from \( f^{-1}(x_h) \) to a vertex in \( A_b \). Such a path can clearly be extended to a path of length \( m + 1 \) in \( L(A_b) \cup f^{-1}(x_h) \subseteq T^* \). The image of that path, denoted by \( Q \), is path of length \( m + 1 \) in \( H \) starting at \( x_b \) and avoiding \( x_{b+1}, \ldots, x_{b-(m+1)} \). Note that \( Q \) has \( m + 2 \) vertices.

Now, we consider two subcases.

**Subcase 2.1.** \( b = 1 \).

In this subcase, we have \( V(P) \cap V(Q) = \{ x_1 \} \) or \( \{ x_0, x_1 \} \), where \( x_0 = x \). In either case, there are \( m \) vertices on \( Q - P \) within distance \( m + 1 \leq t - (m + 1) \) from \( x \). Hence \( B(x) - P \) contains at least \( m \) vertices, and we are done.

**Subcase 2.2.** \( 2 \leq b \leq m \).

By the definition of \( b, \) we have \( x_1, \ldots, x_{b-1} \notin I \). By our earlier discussion, \( x_1 \) has a neighbor \( z_1 \) in \( H \) not on \( P \), while for each \( j \in [2, \ldots, b-1] \), \( x_j \) has a neighbor \( z_j \) in \( H \) not on \( P; z_j \) and \( z_j' \) are at distance \( j + 1 \) and \( j + 2 \) from \( x \), respectively. Since \( j \leq b - 1 \), we have \( j + 2 \leq b + 1 \leq m + 1 \leq t - (m + 1) \). Hence \( z_1, z_2, z_3, \ldots, z_{b-1}, z_{b-1}' \in B(x) - P \).

Let \( q, 0 \leq q \leq b \), denote the smallest index such that \( x_q \) lies on \( Q \), then \( P \cap Q = P[x_q, x_b] \) (see Fig. 1). Recall that \( Q \) has \( m + 2 \) vertices. The first \( (m + 1) - \max \{ q, b - q \} \) vertices on \( Q - P \) are within distance \( m + 1 \leq t - (m + 1) \) from \( x_0 = x \) and hence belong to \( B(x) - P \). If \( q = 1 \), then none of \( z_2, z_2', \ldots, z_{b-1}, z_{b-1}' \) lies on \( Q - P \), so the first \( (m + 1) - (b - 1) \) vertices on \( Q - P \) together with \( z_2, z_2', \ldots, z_{b-1}, z_{b-1}' \) give \( (m + 1) - (b - 1) + 2(b - 2) = m + b - 2 \geq m \) vertices in \( B(x) - P \). If \( q = 0 \) or \( b \), then none of \( z_2, z_2', \ldots, z_{b-1}, z_{b-1}' \) lies on \( Q - P \). These \( 2b - 3 \) vertices together with the first \( (m + 1) - b \) vertices on \( Q - P \) give \( 2b - 3 + (m + 1) - b = m + b - 2 \geq m \) vertices in \( B(x) - P \). So we may assume that \( 2 \leq q \leq b - 1 \), which implies that \( b \geq 3 \). In this case, at most two of \( z_1, z_2, z_2', \ldots, z_{b-1}, z_{b-1}' \) (namely, \( z_q \) and \( z_q' \)) may lie on \( Q - P \) (see Fig. 1). Now, \( 2b - 5 \) of those which are not on \( Q - P \) together with the first \( (m + 1) - \max \{ q, b - q \} \) vertices on \( Q - P \), all
belong to $B(x) - P$. Hence, $|B(x) - P| \geq (2b - 5) + (m + 1) - \max\{q, b - q\}$. Since $(2b - 5) + (m + 1) - q = m + (b - 3) + (b - 1 - q) \geq m$ and $(2b - 5) + (m + 1) - (b - q) = m + (b + q - 4) \geq m + (3 + 2 - 4) > m$, we conclude that $|B(x) - P| \geq m$. This completes our proof.

Note added. Since the initial submission of this paper, two papers on Dobson’s conjecture have appeared. Haxell and Luczak [3] proved Dobson’s conjecture for the case $t \geq 4$ and $k \geq 3t^2 + 2t$ and for the case $t \geq 3[\log_{(k - 1)/t} k] + 1/2k$. See also [6].

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