NOTE

Anti-Ramsey Numbers of Subdivided Graphs

Tao Jiang

Department of Mathematics and Statistics, Miami University,
Oxford, Ohio 45056
E-mail: jiangt@muohio.edu

Received January 23, 2001; published online April 23, 2002

Given a positive integer $n$ and a family $F$ of graphs, the anti-Ramsey number $f(n, F)$ is the maximum number of colors in an edge-coloring of $K_n$ such that no subgraph of $K_n$ belonging to $F$ has distinct colors on its edges. The Turán number $ex(n, F)$ is the maximum number of edges of an $n$-vertex graph that does not contain a member of $F$ as a subgraph. P. Erdős et al. (1975, in Colloq. Math. Soc. Janos Bolyai, Vol. 10, pp. 633–643, North-Holland, Amsterdam) showed for all graphs $H$ that $f(n, H) - ex(n, F) = o(n^2)$, where $F = \{H - e : e \in E(H)\}$. We strengthen their result for the class of graphs in which each edge is incident to a vertex of degree two. We show that $f(n, H) - ex(n, F) = O(n)$ when $H$ belongs to this class. This follows from a new upper bound on $f(n, H)$ that we prove for all graphs $H$ and asymptotically determines $f(n, H)$ for certain graphs $H$.

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1. INTRODUCTION

We consider only nonempty simple graphs. A subgraph of an edge-colored graph is rainbow if all of its edges have different colors. Given a positive integer $n$ and a family $F$ of graphs, the anti-Ramsey number $f(n, F)$ is the maximum number of colors in a coloring of $E(K_n)$ that has no rainbow copy of any graph in $F$. For the purpose of this note, we call a coloring that does not contain a rainbow copy of any graph in $F$ a $F$-free coloring.

Anti-Ramsey numbers were introduced by Erdős et al. [4]. They showed that these are closely related to Turán numbers. The Turán number $ex(n, F)$ of $F$ is the maximum number of edges of an $n$-vertex simple graph having no member of $F$ as a subgraph. Given a coloring $c$ of a host
graph $G$, we define a \textit{representing graph} of $c$ to be a spanning subgraph $L$ of $G$ obtained by taking one edge of each color in $c$ (where $L$ may contain isolated vertices). Given a positive integer $n$ and a graph $H$, clearly a representing graph of an $H$-free coloring of $E(K_n)$ does not contain $H$ as a subgraph. Thus we have $f(n, H) \leq ex(n, H)$. Let $\mathcal{H} = \{H - e : e \in E(H)\}$.

Let $G$ be a subgraph of $K_n$ with $ex(n, \mathcal{H})$ edges that does not contain any member of $\mathcal{H}$ as a subgraph. We can define an $H$-free coloring of $E(K_n)$ using at least $ex(n, \mathcal{H})$ colors by coloring the edges of $G$ with distinct colors and then coloring the remaining edges (if any) in $K_n$ with a new color. Hence, $f(n, H) \geq ex(n, \mathcal{H})$.

\textbf{Proposition 1.1.} Given a positive integer $n$ and a graph $H$, we have

$$ex(n, \mathcal{H}) \leq f(n, H) \leq ex(n, H),$$

where $\mathcal{H} = \{H - e : e \in E(H)\}$.

The lower and upper bound in Proposition 1.1 could differ even in the order of magnitude. For instance, when $H$ is an odd cycle, $ex(n, H)$ is quadratic in $n$ while $ex(n, \mathcal{H})$ is linear in $n$. In general, the upper bound $ex(n, H)$ is quite loose, and $f(n, H)$ is much closer to $ex(n, \mathcal{H})$. Erdős et al. [4] showed that $f(n, H) \leq ex(n, H) + o(n^2)$ as $n \rightarrow \infty$. Thus we have

\textbf{Theorem A [4].} $f(n, H) - ex(n, \mathcal{H}) = o(n^2)$, as $n \rightarrow \infty$.

If $d = \min \{\chi(G) : G \in \mathcal{H}\} \geq 3$, then by an earlier result of Erdős and Simonovits [5], we have $ex(n, \mathcal{H}) = \frac{2}{d-1}(\frac{1}{2}n^2) + o(n^2)$, and Theorem A yields $f(n, H) = \frac{2}{d-1}(\frac{1}{2}n^2) + o(n^2)$. This determines $f(n, H)$ asymptotically. If $d \leq 2$, however, we have $ex(n, \mathcal{H}) = o(n^2)$, and Theorem A says little about $f(n, H)$. Erdős et al. [4] therefore proposed studying $f(n, H)$ for graphs $H$ that contains an edge whose deletion leaves a bipartite subgraph, and they put forward two conjectures about $f(n, H)$ when $H$ is a path or a cycle.

Simonovits and Sós [9] proved the conjecture for paths, showing for large $n$ that $f(n, P_{2t+3+\varepsilon}) = tn - \binom{t+1}{2} + 1 + \varepsilon$, where $\varepsilon = 0, 1$ and $P_t$ is a path on $t$ vertices. Jiang and West [7] considered $f(n, T)$ when $T$ is a general tree of a given size. For cycles, Erdős et al. [4] conjectured that for every fixed $k \geq 3$ $f(n, C_k) = n(\frac{k-2}{2} + \frac{1}{k-1}) + O(1)$, and they obtained a $C_k$-free coloring of $E(K_n)$ using the conjectured number of colors. They noted that the conjecture holds for $k = 3$. Alon [1] proved the conjecture for $k \leq 4$ and proved that $f(n, C_k) \leq n(k-2) + \binom{k+1}{2}$ in general. Jiang and West [8] proved the conjecture for $k \leq 6$ and improved the general upper bound to $f(n, C_k) \leq n(\frac{k+1}{2} - \frac{1}{k-1}) - (k-2)$ for all $k$ and to $f(n, C_k) \leq nk/2 - (k-2)$ when $k$ is even. Axenovich and Jiang [3] initiated the study.
of the anti-Ramsey numbers for complete bipartite graphs. They showed for all $t \geq 3$ that $f(n, K_{2, t}) = \sqrt{t-2} \cdot n^{3/2} + O(n^{3/2})$ by proving that $f(n, K_{2, t}) - ex(n, K_{2, t-1}) = O(n)$.

Note that in the cases mentioned above when $H$ is a path, a cycle, or a complete bipartite graph with one bipartite set of size 2, one has $f(n, H) - ex(n, H) = O(n)$. In this note, we establish a more general fact that if $H$ is a graph in which each edge is incident to a vertex of degree two then $f(n, H) - ex(H) = O(n)$ always holds (which immediately implies the result obtained in [3]). In particular, this applies to graphs $H$ obtained by subdividing each edge of any given graph $G$ at least once. The claim follows from the following upper bound on $f(n, H)$ that holds for all (nonempty) graphs $H$.

**Theorem 1.2.** Given a graph $H$, let $H' = \{H - v : v \in V(H), d_H(v) = 2\}$. Suppose $H$ has $p$ vertices and $q$ edges. For all positive integers $n$, we have

$$f(n, H) \leq ex(n, H') + bn,$$

where $b = \max\{2p - 2, q - 2\}$.

Now suppose $H$ is a (nonempty) graph in which each edge is incident to a vertex of degree two. Let $e$ be any edge in $H$. By our assumption, $e$ is incident to a vertex $v$ of degree two in $H$. Note that $H - e$ contains $H - v$ as a subgraph. This shows that every member of $H$ contains a subgraph that is in $H'$. Thus we have $ex(n, H') \leq ex(n, H)$. This observation together with Theorem 1.2 and Proposition 1.1 yields

**Theorem 1.3.** Let $H$ be a graph in which each edge is incident to a vertex of degree two. Suppose $H$ has $p$ vertices and $q$ edges. Let $H = \{H - e : e \in E(H)\}$ and $b = \max\{2p - 2, q - 2\}$. We have

$$ex(n, H) \leq f(n, H) \leq ex(n, H) + bn.$$ 

Hence $f(n, H) - ex(n, H) = O(n)$, as $n \to \infty$.

It is known that $ex(n, G)$ grows at least super linearly in $n$ for any graph $G$ which is not a forest. Hence Theorem 1.3 implies

**Corollary 1.4.** If $H$ is a graph containing at least two cycles in which each edge is incident to a vertex of degree two, then

$$f(n, H) = ex(n, H)(1 + o(1)),$$

where $H = \{H - e : e \in (H)\}$. 

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For the rest of the paper, we give a proof of Theorem 1.2. Given a graph $G$ and a subset $U \subseteq V(G)$, we use $G[U]$ to denote the subgraph of $G$ induced by $U$. Given a vertex $u$ in $G$, $N_G(u)$ denotes its neighborhood in $G$.

2. PROOF OF THEOREM 1.2

Let $H$ be a given graph, and let $\mathcal{H}_2 = \{H - v : v \in V(H), d_H(v) = 2\}$. Suppose $H$ has $p$ vertices and $q$ edges. Then each graph in $\mathcal{H}_2$ has $p-1$ vertices and $q-2$ edges. We introduce some notions for convenience. Given any graph $D \in \mathcal{H}_2$, by definition, $D = H - v$ for some vertex $v$ of degree two in $H$. We use $a(D)$ and $b(D)$ to denote the two neighbors of $v$ in $H$, and call them the two ends of $D$. Let $S(D) = \{a(D), b(D)\}$.

Vertices $u_1, u_{k+1}$ are the two ends of $R$. For $k \geq 2$, if in the above definition, $u_1, \ldots, u_k$ are distinct and $u_{k+1} = u_1$, then $R$ is an $\mathcal{H}_2$-ring of length $k$.

Lemma 2.1. Let $G$ be a graph on $n$ vertices with more than $\text{ex}(n, \mathcal{H}_2) + (q-2)(n-1)$ edges. Then $G$ contains an $\mathcal{H}_2$-ring.

Proof. Recall that each graph in $\mathcal{H}_2$ has $q-2$ edges. Let $\mathcal{D}$ be a maximal collection of pairwise edge-disjoint subgraphs of $G$ which belong to $\mathcal{H}_2$. Suppose $\mathcal{D}$ contains $m$ members. By the maximality of $\mathcal{D}$, $G - E(\mathcal{D})$ contains no subgraphs that belong to $\mathcal{H}_2$. Hence we have $e(G - E(\mathcal{D})) < \text{ex}(n, \mathcal{H}_2)$. Thus, $e(G) < \text{ex}(n, \mathcal{H}_2) + m(q-2)$. Since $e(G) > \text{ex}(n, \mathcal{H}_2) + (q-2)(n-1)$, it follows that $m > n-1$. Now, construct a graph $F$ with $V(F) = V(G)$ as follows. For each member $D$ (which is a graph in $\mathcal{H}_2$) of $\mathcal{D}$, where $S(D) = \{u, v\}$, we include $uv$ as an edge in $F$. Since $\mathcal{D}$ has $m$ members, the resulting graph $F$ is an $n$-vertex loopless multigraph with $m > n-1$ edges. Hence $F$ contains a cycle $C$. The union of the members of $\mathcal{D}$ which correspond to the edges $C$ forms an $\mathcal{H}_2$-ring in $G$.

A graph $T$ obtained from an $\mathcal{H}_2$-string $R$ of length $k$ by adding a new vertex $x$ not in $R$ and making it adjacent to the two ends of $R$ is an $\mathcal{H}_2$ string-tie of length $k$. Note that $H$ is an $\mathcal{H}_2$-string-tie of length 1.

Lemma 2.2. Let $c$ be a coloring of $E(K_n)$ that contains a rainbow $\mathcal{H}_2$-string-tie. Then $c$ contains a rainbow copy of $H$.

Proof. Let $T$ be a rainbow $\mathcal{H}_2$-string-tie in $c$ of minimum length. Suppose $T$ is obtained from an $\mathcal{H}_2$-string $R$ of length $k$ by adding a vertex $x$ not in $R$ and making it adjacent to the two ends of $R$. Suppose $R$ is the edge-disjoint union of $D_1, \ldots, D_k$, where $D_i \in \mathcal{H}_2$, and $S(D_i) = \{u_i, u_{i+1}\}$ for
all \( i \in [k] \). If \( k = 1 \) then \( T \) is a rainbow \( H \). So we may assume \( k \geq 2 \). Let \( T_1 = D_1 \cup xu_1 \) and \( T_2 = D_2 \cup \cdots \cup D_k \cup xu_{k+1} \). Since \( T \) is rainbow, the color \( c(xu_2) \) cannot be used in both \( T_1 \) and \( T_2 \). Now \( xu_2 \) completes a rainbow \( \mathcal{H} \)-string-tie with either \( T_1 \) or \( T_2 \), which is shorter than \( T \), a contradiction. 

**Lemma 2.3.** Suppose \( c \) is an \( H \)-free coloring of \( E(K_n) \) and \( R \) is a rainbow \( \mathcal{H} \)-ring in \( c \). Let \( x \in V(K_n) - V(R) \). Suppose there exists \( y \in S(R) \) such that the color \( c(xy) \) is not used on the edges of \( R \), then \( c(xy') = c(xy) \) for all \( y' \in S(R) \).

**Proof.** Otherwise, suppose there exists \( y' \in S(R) \) such that \( c(xy') \neq c(xy) \). Vertices \( y \) and \( y' \) partition \( R \) into two \( \mathcal{H} \)-strings \( R_1, R_2 \) sharing \( y, y' \) as common ends. Since \( R \) is rainbow, one of \( R_1 \) and \( R_2 \) avoids the color \( c(xy') \). Suppose \( R_1 \) does. Now \( R_1 \cup \{xy, xy'\} \) is a rainbow \( \mathcal{H} \)-string-tie, and by Lemma 2.2, \( c \) contains a rainbow copy of \( H \), a contradiction. 

Now we are ready to prove Theorem 1.2.

**Proof of Theorem 1.2.** We use induction on \( n \), with the claim holding trivially for small values of \( n \). Let \( c \) be an \( H \)-free coloring of \( E(K_n) \) using \( f(n, H) \) colors. Let \( L \) be a representing graph of \( c \). If \( L \) contains no \( \mathcal{H} \)-ring, then by Lemma 2.1, we have \( f(n, H) = e(L) \leq ex(n, \mathcal{H}) + (q-2)(n-1) \leq ex(n, \mathcal{H}) + bn \), recalling that \( b = \max\{2p-2, q-2\} \). So we may assume that \( L \) contains an \( \mathcal{H} \)-ring \( R \) of length \( k \), where \( k \geq 2 \). Since \( c \) contains no rainbow \( H \), by Lemma 2.2, \( L \) contains no \( \mathcal{H} \)-string-tie.

**Claim 2.4.** The number of edges in \( L[V(R)] \) that are incident to \( S(R) \) is at most \((p-1)k\).

**Proof of Claim 2.4.** Suppose \( R \) consists of \( D_1, \ldots, D_k \), with \( D_i \in \mathcal{H}_2 \), and \( S(D_i) = \{u_i, u_{i+1}\} \) for \( i \in [k] \) (with indices taken modulo \( k \)). Let \( v \in V(R) \). Suppose \( N_L(v) \cap S(R) = \{u_{j_1}, u_{j_2}, \ldots, u_{j_m}\} \), where \( j_1 < j_2 < \cdots < j_m \). For each \( i \in [m] \), let \( F_i = \bigcup_{l}^{j_{i+1} - 1} D_l \) (with indices \( l \) taken modulo \( k \)). If \( v \notin V(F_i) \) for some \( i \in [m] \) then \( F_i \cup \{vu_{j_1}, vu_{j_{m-1}}\} \) would form an \( \mathcal{H} \)-string-tie in \( L \), a contradiction. Hence \( v \in V(F_i) \) for all \( i \in [m] \). So, in particular, \( v \) is contained in at least \( m \) of the \( D_i ' \)s. Hence we have

\[
\begin{align*}
\# \text{ edges in } L[V(R)] \text{ incident to } S(R) & \leq \sum_{v \in V(R)} \# \text{ edges in } L \text{ between } v \text{ and } S(R) \\
& \leq \sum_{v \in V(R)} |\{i \in [k] : v \in V(D_i)\}| \\
& = \sum_{i=1}^{k} |V(D_i)| = k(p-1).
\end{align*}
\]
Now, let \( K = K_n \), and let \( K' = K - \{ u_1, \ldots, u_{k-1} \} \). Consider any color \( \alpha \) that is used by \( c \) in \( K \) but not in \( K' \). By the definition of \( L \) as a representing graph of \( c \), there is an edge \( e \) of \( L \) such that \( c(e) = \alpha \). Since \( \alpha \) is not used in \( K' \), one of the endpoints of \( e \) must be in \( \{ u_1, \ldots, u_{k-1} \} \). Suppose \( e = xu_i \), where \( i \in [k-1] \). Suppose \( x \notin V(R) \). Then \( xu_i \) does not lie in \( R \) and therefore \( c(xu_i) \) is not used on the edges of \( R \) (recall that edges of \( L \) have distinct colors). By Lemma 2.3, we have \( \alpha = c(xu_i) = c(xu_k) \), contradicting our assumption that \( \alpha \) is not used in \( K' \) (note that \( xu_k \in E(K') \)). Hence \( x \in V(R) \), and \( \alpha = c(xu_i) \) is used on an edge of \( L[V(R)] \) that is incident to \( S(R) \). By Claim 2.4, there are at most \( (p-1)/k \) such colors. Now, since \( K' \) is a complete graph of order \( n-k+1 \), and \( c \) restricted to \( K' \) is \( H \)-free, by induction hypothesis we have

\[
f(n, H) - (p-1)k \leq \# \text{ colors used by } c \text{ in } K'
\]

\[
\leq ex(n-k+l, H_2) + bn - (p-1)k \leq ex(n, H_2) + bn,
\]

recalling that \( b = \max\{2p-2, q-2\} \) and \( k \geq 2 \). This completes the proof of Theorem 1.2. \( \square \)

ACKNOWLEDGMENT

The author is indebted to the referee for helpful comments.

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